

# CONTROLLABILITY OF THE IMPULSIVE SEMILINEAR BEAM EQUATION WITH MEMORY AND DELAY

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**ABSTRACT.** The semilinear beam equation with impulses, memory and delay is considered. We obtain the approximate controllability. This is done by employing a technique that avoids fixed point theorems and pulling back the control solution to a fixed curve in a short time interval. Demonstrating, once again, that the controllability of a system is robust under the influence of impulses and delays.

## 1. INTRODUCTION

Beams have been used since ancient times in bridges, buildings, for example. Through the millennia, understanding the dynamics and controllability of beams, including bending and vibration, has been of great importance. One of the pioneers in studying them was Galileo presenting how they should be approached, however, it was not until the late 17th century with the progress of elasticity that Leonhard Euler and Daniel Bernoulli provided a second-order spatial derivatives mathematical model that later, in 1921, Stephen Timoshenko improved by including a shear deformation and rotational inertia effects, obtaining fourth order mathematical model (see [16, 17, 18] for details).

In particular, the impulsive semilinear beam equations of the form (1.1) is of greatly interest in mechanical engineer and nanotechnology design [19, 20], and the memory and delay are characterized by the viscoelasticity property and response of the materials. In this paper, we are exploring the approximate controllability of

$$(1.1) \quad w_{tt} - 2\beta\Delta w_t + \Delta^2 w = u(t, x) + f(t, w(t-r), w_t(t-r), u) + \int_0^t M(t-s)g(w(s-r, x))ds,$$

$t \neq t_k$ , subjected to the initial-boundary conditions and impulses

$$(1.2) \quad \begin{cases} w(t, x) = \Delta w(t, x) = 0, & \text{in } (0, \tau) \times \partial\Omega, \\ w(s, x) = \phi_1(s, x), & \text{in } [-r, 0] \times \Omega, \\ w_t(s, x) = \phi_2(s, x), & \text{in } [-r, 0] \times \Omega, \\ w_t(t_k^+, x) = w_t(t_k^-, x) + I_k(t_k, w(t_k, x), w_t(t_k, x), u(t_k, x)), & k = 1, \dots, p, \end{cases}$$

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where  $\Omega \subseteq \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded domain, the constant  $\beta > 1$  and the real-valued functions  $w = w(t, x)$  in  $(0, \tau] \times \Omega$  denotes the deflection of a beam,  $u$  in  $(0, \tau] \times \Omega$  represents the distributed control,  $M$  acts as convolution kernel with respect to the time variable,  $I_k$  is defined on  $[0, \tau] \times \mathbb{R}^3$  and  $g$  on  $\mathbb{R}$ ,  $f$  on  $[0, \tau] \times \mathbb{R}^3$  are the nonlinearities. Under the assumption that  $M \in L^\infty((0, \tau) \times \Omega)$  and  $g, f, I_k$  are smooth enough so that, for all  $\phi, \psi \in \mathcal{C}([-r, 0], L^2(\Omega))$  and  $u \in L^2([0, \tau]; L^2(\Omega))$  the equation (1.1) admits only one mild solution on  $[-r, \tau]$  and

$$(1.3) \quad |f(t, y, v, u)| \leq a\sqrt{|y|^2 + |v|^2} + b,$$

with  $t \in [0, \tau]$ ,  $y, v, u \in \mathbb{R}$ ,  $a_0, b_0 \geq 0$ .

This article has been inspired by the series of papers from Carrasco, Leiva, Merentes and Sanchez [7, 8, 6] on the approximate controllability of semilinear beam and the works of Guevara and Leiva [11, 12] on the approximate controllability for the semilinear heat and strongly damped wave equations with memory and delays. Here we prove the approximate controllability of the beam equation (1.1) under the initial-boundary condition (1.2) with memory, impulses and delay terms by applying A.E. Bashirov, N. Ghahramanlou, N. Mahmudov, N. Semi and H. Etikan technique [1, 2, 5], and avoiding the Rothe's fixed point theorem used in [7, 6] and the Schauder fixed point theorem applied in [8].

The structure of this paper is as follow: In section 2, we present the abstract formulation of the beam equation (1.1). Section 3, recalls the linear controllability characterization of the problem. In section 4, the approximated controllability of the beam equation with memory, delay and impulses is proved.

## 2. ABSTRACT FORMULATION OF THE PROBLEM

In this section we choose the appropriate Hilbert space where the Cauchy problem (1.1)-(1.2) can be written as an abstract differential equation.

First of all, notice that the term  $-2\beta\Delta w_t$  in the equation (1.1) acts as a damping force, thus the energy space used to set up the wave equation is not suitable here. Even so, in [15], Oliveira shows that the uncontrolled linear equation can be transformed into a system of parabolic equations of the form  $w_t = D\Delta w$ , obtaining that corresponding space for the abstract formulation of the problem is  $\mathcal{Z}^1 = [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$  and proving that the linear part of this system generates a strongly continuous analytic semigroup in this space.

Consider the Hilbert space  $\mathcal{X} = L^2(\Omega)$ , and denote  $\mathcal{A} = -\Delta$  with the eigenvalues  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \rightarrow \infty$ , each one with multiplicity  $\gamma_j < \infty$  equal to its corresponding eigenspace dimension. Recall,  $\mathcal{A}$  satisfies the following properties:

- (i) There exists a complete orthonormal set  $\{\phi_{j_k}\}$  of eigenvectors of  $\mathcal{A}$ .

(ii) For all  $x \in D(\mathcal{A})$ ,

$$\mathcal{A}x = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle \xi, \phi_{jk} \rangle \phi_{jk} = \sum_{j=1}^{\infty} \lambda_j E_j x,$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{X}$ ,  $E_n x = \sum_{k=1}^{\gamma_j} \langle z, \phi_{jk} \rangle \phi_{jk}$ , and  $\{E_j\}$  is a family of complete orthogonal projections in  $\mathcal{X}$ .

(iii)  $-\mathcal{A}$  generates an analytic semigroup  $\{S(t)\}_{t \geq 0}$  given by

$$S(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x \quad \text{and} \quad \|S(t)\| \leq e^{-\lambda_1 t}.$$

(iv) For  $\alpha \geq 0$  the fractional powered spaces  $\mathcal{X}^\alpha$  are given by

$$\mathcal{X}^\alpha = D(\mathcal{A}^\alpha) = \left\{ x \in \mathcal{X} : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \|E_j x\|^2 < \infty \right\}$$

equipped with the norm  $\|x\|_\alpha^2 = \|\mathcal{A}^\alpha x\|^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \|E_j x\|^2$ , where  $\mathcal{A}^\alpha x = \sum_{j=1}^{\infty} \lambda_j^\alpha E_j x$ .

In particular, taking  $\alpha = 1$ , the Hilbert space  $\mathcal{Z}^1 = \mathcal{X}^1 \times \mathcal{X}$  has the norm

$$\left\| \begin{pmatrix} w \\ v \end{pmatrix} \right\|_{\mathcal{Z}^1}^2 = \|w\|_1^2 + \|v\|^2.$$

Using the above notation, we rewrite the system (1.1)-(1.2) as the second-order ordinary differential equations in the Hilbert space  $\mathcal{X}$

$$(2.4) \quad \begin{cases} w''(t) = -\mathcal{A}^2 w(t) - 2\beta \mathcal{A} w'(t) + u(t) + \int_0^t M(t,s) g^e(w(s-r)) ds & t > 0, t \neq t_k, \\ \quad + f^e(t, w(t-r), w'(t-r), u(t)), \\ w(s) = \phi_1(s), \quad w'(s) = \phi_2(s), & s \in [-r, 0], \\ w'(t_k^+) = w'(t_k^-) + I_k^e(t_k, w(t_k), w'(t_k), u(t_k)), & k = 1, \dots, p, \end{cases}$$

where  $\mathcal{U} = \mathcal{X} = L^2(\Omega)$ , and

$$\begin{aligned} I_k^e : [0, \tau] \times \mathcal{Z}^1 \times \mathcal{U} &\longrightarrow \mathcal{X} \\ (t, w, v, u)(\cdot) &\longmapsto I_k(t, w(\cdot), v(\cdot), u(\cdot)), \end{aligned}$$

$$\begin{aligned} f^e : [0, \tau] \times \mathcal{C}(-r, 0; \mathcal{Z}^1) \times \mathcal{U} &\longrightarrow \mathcal{X} \\ (t, \Phi, u)(\cdot) &\longmapsto f(t, \phi_1(-r, \cdot), \phi_2(-r, \cdot), u(\cdot)), \end{aligned}$$

and

$$\begin{aligned} g^e : \mathcal{C}(-r, 0; \mathcal{Z}^1) &\longrightarrow \mathcal{Z}^1 \\ \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &\longmapsto g(\phi_1(\cdot - r)). \end{aligned}$$

Finally, by changing variables,  $v = w'$ , the systems (2.4) can be written as an abstract first order functional differential equations with memory and impulses in  $\mathcal{Z}^1$

$$(2.5) \quad \begin{cases} z' = -\mathbb{A}z + \mathbb{B}u + \int_0^t \mathbb{M}_g(t, s, z_s(-r))ds + \mathbb{F}(t, z_t(-r), u(s)), & z \in \mathcal{Z}^1, t \geq 0, \\ z(s) = \Phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathbb{I}_k(t_k, z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p, \end{cases}$$

where  $\mathbb{A} = \begin{pmatrix} 0 & I_{\mathcal{X}} \\ -\mathcal{A}^2 & -2\beta\mathcal{A} \end{pmatrix}$  is a unbounded linear operator with domain

$$D(\mathbb{A}) = \{w \in H^4(\Omega) : w = \Delta w = 0\} \times D(\mathcal{A}),$$

with  $I_{\mathcal{X}}$  the identity in  $\mathcal{X}$ ,  $z = (w, v)^T$ ,  $u \in L^2(0, \tau; \mathcal{U})$ ,  $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{C}(-r, 0; \mathcal{Z}^1)$ ,  $\mathbb{B} : \mathcal{U} \longrightarrow$

$\mathcal{Z}^1$  is the bounded linear operator defined by  $\mathbb{B}u = \begin{pmatrix} 0 \\ u \end{pmatrix}$ , and the functions

$$(2.6) \quad \begin{aligned} \mathbb{I}_k : [0, \tau] \times \mathcal{Z}^1 \times \mathcal{U} &\longrightarrow \mathcal{Z}^1 \\ (t, z, u) &\longmapsto \begin{pmatrix} 0 \\ I_k^e(t, w, v, u) \end{pmatrix} \\ \mathbb{F} : [0, \tau] \times \mathcal{C}(-r, 0; \mathcal{Z}^1) \times \mathcal{U} &\longrightarrow \mathcal{Z}^1 \\ (t, \Phi, u) &\longmapsto \begin{pmatrix} 0 \\ f^e(t, \phi_1(-r), \phi_2(-r), u) \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{M}_g : [0, \tau] \times [0, \tau] \times \mathcal{C}(-r, 0; \mathcal{Z}^1) &\longrightarrow \mathcal{Z}^1 \\ (t, s, \Phi) &\longmapsto \begin{pmatrix} 0 \\ M(t, s)g^e(\Phi) \end{pmatrix}. \end{aligned}$$

Moreover, this abstract formulation together with condition (1.3) and the continuous imbedding  $\mathcal{X}^1 \subset \mathcal{X}$  yields

**PROPOSITION 2.1.** *There exist constants  $\tilde{a}, \tilde{b} > 0$  such that, for all  $(t, \Phi, u) \in [0, \tau] \times \mathcal{C}(-r, 0; \mathcal{Z}^1) \times \mathcal{U}$  the following inequality holds*

$$(2.7) \quad \|\mathbb{F}(t, \Phi, u)\|_{\mathcal{Z}^1} \leq \tilde{a}\|\Phi(-r)\|_{\mathcal{Z}^1} + \tilde{b}.$$

In [7] is proved that the linear unbounded operator  $\mathbb{A}$  generates a strongly continuous compact semigroup in the space  $\mathcal{Z}^1$  which decays exponentially to zero, precisely:

PROPOSITION 2.2. *The operator  $\mathcal{A}$  is the infinitesimal generator of a strongly continuous compact semigroup  $\{T(t)\}_{t \geq 0}$  represented by*

$$(2.8) \quad T(t)z = \sum_{j=1}^{\infty} e^{\mathbb{A}_j t} P_j z, \quad z \in Z_1, \quad t \geq 0,$$

where  $\{P_j\}_{j \geq 0}$  is a complete family of orthogonal projections in the Hilbert space  $Z_1$  given by

$$(2.9) \quad P_j = \text{diag}(E_j, E_j),$$

and

$$\mathbb{A}_j = K_j P_j, \quad K_j = \begin{pmatrix} 0 & 1 \\ -\lambda_j^2 & -2\beta\lambda_j \end{pmatrix} \quad j \geq 1,$$

and there exists  $M \geq 1$  and  $\mu > 0$  such that

$$\|T(t)\| \leq M e^{-\mu t}, \quad t \geq 0,$$

### 3. APPROXIMATE CONTROLLABILITY OF THE LINEAR SYSTEM

This section is devoted to characterize the approximate controllability of the linear system. Thus, for all  $z_0 \in \mathcal{Z}^1$  and  $u \in L^2([0, \tau]; \mathcal{U})$  consider the initial value problem

$$(3.10) \quad \begin{cases} z'(t) = \mathbb{A}z(t) + \mathbb{B}u(t), \\ z(t_0) = z_0, \end{cases}$$

obtained from (2.5). It admits only one mild solution on  $0 \leq t_0 \leq t \leq \tau$  given by

$$(3.11) \quad z(t) = T(t - t_0)z_0 + \int_{t_0}^t T(t - s)\mathbb{B}u(s)ds; \quad t \in [t_0, \tau], \quad 0 \leq t_0 \leq \tau.$$

DEFINITION 3.1. (**Approximate Controllability of (3.10)**) *The system (3.10) is said to be approximately controllable on  $[t_0, \tau]$  if for every  $z_0, z_1 \in Z$ ,  $\varepsilon > 0$  there exists  $u \in L_2(t_0, \tau; U)$  such that the solution  $z(t)$  of (4.14) corresponding to  $u$  verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

For the system (3.10) and  $\tau > 0$ , we have the following notions:

(1)  $G_{\tau\delta}$  is the controllability operator defined by

$$G_{\tau\delta} : L^2(\tau - \delta, \tau; \mathcal{U}) \longrightarrow \mathcal{Z}^1$$

$$u \longmapsto \int_{\tau-\delta}^{\tau} T(\tau - s)\mathbb{B}u(s)ds,$$

with corresponding adjoint  $G_{\tau\delta}^*$  given by

$$G_{\tau\delta}^* : \mathcal{Z}^1 \longrightarrow L^2(\tau - \delta, \tau; \mathcal{U})$$

$$z \longmapsto \mathbb{B}^* T^*(\tau - \cdot)z.$$

(2) The Gramian controllability operator is

$$Q_{\tau\delta*} = G_{\tau\delta}G_{\tau\delta}^* = \int_{\tau-\delta}^{\tau} T(\tau-t)\mathbb{B}\mathbb{B}^*T^*(\tau-t)dt.$$

In general, for linear bounded operator  $G$  between Hilbert spaces  $\mathcal{W}$  and  $\mathcal{Z}$ , the following lemma holds (see [3, 4, 14]).

LEMMA 3.1. *The approximate controllability of the linear system (3.10) on  $[\tau-\delta, \tau]$  is equivalent to any of the following statements*

- (a)  $\overline{\text{Rang}(G_{\tau\delta})} = \mathcal{Z}^1$ .
- (b)  $\ker(G_{\tau\delta}^*) = 0$ .
- (c) For  $0 \neq z \in \mathcal{Z}^1$ ,  $\langle Q_{\tau\delta}z, z \rangle > 0$ .

The controllability of the linear system (3.10) on  $[0, \tau]$  was proved by A. Carrasco and H. Leiva in [7]. Theorem 3.1 and Lemma 3.2 characterized the controllability of the system (3.10), their proofs and details can be found in [3, 4, 9, 10, 14]

THEOREM 3.1. *The system (3.10) is approximately controllable on  $[0, \tau]$  if and only if any one of the following conditions hold:*

- (1)  $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + Q_{\tau\delta}^*)^{-1}z = 0$ .
- (2) If  $z \in \mathcal{Z}^1$ ,  $0 < \alpha \leq 1$  and  $u_\alpha = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}z$ , then

$$G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}z \quad \text{and} \quad \lim_{\alpha \rightarrow 0} G_{\tau\delta}u_\alpha = z.$$

Moreover, for each  $v \in L^2([\tau-\delta, \tau]; \mathcal{U})$ , the sequence of controls

$$u_\alpha = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}z + (v - G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}G_{\tau\delta}v),$$

satisfies

$$G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta}^*)^{-1}(z - G_{\tau\delta}v) \quad \text{and} \quad \lim_{\alpha \rightarrow 0} G_{\tau\delta}u_\alpha = z,$$

with the error  $E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}(z + G_{\tau\delta}v)$ ,  $\alpha \in (0, 1]$ .

Theorem 3.1 indicates that the family of linear operators  $\Gamma_{\tau\delta} = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}$  is an approximate right inverse for the  $G_{\tau\delta}$ , in the sense that

$$\lim_{\alpha \rightarrow 0} G_{\tau\delta}\Gamma_{\tau\delta} = I,$$

in the strong topology.

LEMMA 3.2.  $Q_{\tau\delta} > 0$ , if and only if, the linear system (3.10) is controllable on  $[\tau-\delta, \tau]$ . Moreover, for given initial state  $y_0$  and final state  $z_1$ , there exists a sequence of controls  $\{u_\alpha^\delta\}_{0 < \alpha \leq 1}$  in the space  $L^2(\tau-\delta, \tau; \mathcal{U})$ , defined by

$$u_\alpha = u_\alpha^\delta = G_{\tau\delta}^*(\alpha I + G_{\tau\delta}G_{\tau\delta}^*)^{-1}(z_1 - T(\tau)y_0),$$

such that the solutions  $y(t) = y(t, \tau - \delta, y_0, u_\alpha^\delta)$  of the initial value problem

$$(3.12) \quad \begin{cases} y' = \mathbb{A}y + \mathbb{B}u_\alpha(t), & y \in \mathcal{Z}^1, \quad t > 0, \\ y(\tau - \delta) = y_0, \end{cases}$$

satisfies

$$(3.13) \quad \lim_{\alpha \rightarrow 0^+} y(\tau) = \lim_{\alpha \rightarrow 0^+} \left( T(\delta)y_0 + \int_{\tau-\delta}^{\tau} T(\tau-s)\mathbb{B}u_\alpha(s)ds \right) = z_1.$$

#### 4. CONTROLLABILITY OF THE SEMILINEAR SYSTEM

This section is devoted to prove the main result of this paper, the approximate controllability of the beam equation Theorem 4.1, which is it is equivalent to prove the controllability of the abstract system (2.5) given in the following definition

**DEFINITION 4.1. (Approximate Controllability)** *The system (2.5) is said to be approximately controllable on  $[0, \tau]$  if for every  $\epsilon > 0$ , every  $\Phi \in \mathcal{C}(-r, 0; \mathcal{Z}^1)$  and a given initial state  $z_1 \in \mathcal{Z}^1$  there exists  $u \in L^2(0, \tau; \mathcal{U})$ , such that, the corresponding mild solution*

$$(4.14) \quad \begin{aligned} z^u(t) = & T(t)\Phi(0) + \int_0^t T(t-s) \left[ \mathbb{B}u(s) + \left( \int_0^s \mathbb{M}_g(s, l, z(l-r))dl \right) \right] ds \\ & + \int_0^t T(t-s)\mathbb{F}(s, z(s-r), u(s))ds + \sum_{0 < t_k < t} T(t-t_k)\mathbb{I}_k(t_k, z(t_k), u(t_k)), \end{aligned}$$

satisfies  $z(0) = \Phi(0)$  and

$$(4.15) \quad \|z^u(\tau) - z_1\|_{\mathcal{Z}^1} < \epsilon.$$

The approach to obtain (4.15) consist in construct a sequence of controls conducting the system from the initial condition  $\Phi$  to a small ball around  $z_1$ , taking advantage of the delay, which allows us to pullback the corresponding family of solutions to a fixed trajectory in short time interval. Now, we are ready to present the proof of our main result

**THEOREM 4.1.** *Under the condition (2.7) the impulsive semilinear beam equation with memory and delay (1.1) is approximately controllable on  $[0, \tau]$ .*

**Proof.** Let  $\epsilon > 0$ , and given  $\Phi \in \mathcal{C}$  and a final state  $z_1$ . Consider any  $u \in L^2([0, \tau]; \mathcal{U})$  and the corresponding mild solution (4.14) of the initial value problem (2.5), denoted by  $z(t) = z(t, 0, \Phi, u)$ . For  $0 \leq \alpha \leq 1$ , define the control  $u_\alpha^\delta \in L^2([0, \tau]; \mathcal{U})$  as follows

$$u_\alpha^\delta(t) = \begin{cases} u(t), & 0 \leq t \leq \tau - \delta, \\ u_\alpha(t), & \tau - \delta \leq t \leq \tau, \end{cases}$$

with  $u_\alpha = \mathbb{B}^*T^*(\tau - t)(\alpha I + G_{\tau\delta}G_{\tau\delta}^*)^{-1}(z_1 - T(\delta)z(\tau - \delta))$ . Thus,

$0 < \delta < \tau - t_p$  and the corresponding mild solution at time  $\tau$  can be written as follows:

$$\begin{aligned}
z^{\delta,\alpha}(\tau) &= T(\tau)\Phi(0) + \int_0^\tau T(\tau-s) \left[ \mathbb{B}u_\alpha^\delta(s) + \int_0^s \mathbb{M}_g(z^{\delta,\alpha}(l-r))dl \right] ds + \\
&\quad + \int_0^\tau T(\tau-s) \mathbb{F}(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s)) ds + \sum_{0 < t_k < \tau} T(\tau-t_k) \mathbb{I}_k(t_k, z^{\delta,\alpha}(t_k), u_\alpha^\delta(t_k)) \\
&= T(\delta) \left\{ T(\tau-\delta)\Phi(0) + \int_0^{\tau-\delta} T(\tau-\delta-s) \left( \mathbb{B}u_\alpha^\delta(s) + \mathbb{F}(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s)) \right) ds \right. \\
&\quad + \int_0^{\tau-\delta} T(\tau-\delta-s) \int_0^s \mathbb{M}_g(s, l, z^{\delta,\alpha}(l-r)) dl ds \\
&\quad \left. + \sum_{0 < t_k < \tau-\delta} T(\tau-\delta-t_k) \mathbb{I}_k(t_k, z^{\delta,\alpha}(t_k), u_\alpha^\delta(t_k)) \right\} + \\
&\quad + \int_{\tau-\delta}^\tau T(\tau-s) \left( \mathbb{B}u_\alpha(s) + \mathbb{F}(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s)) + \int_0^s \mathbb{M}_g(s, l, z^{\delta,\alpha}(l-r)) dl \right) ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
z^{\delta,\alpha}(\tau) &= T(\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau T(\tau-s) \left( \mathbb{B}u_\alpha(s) + \mathbb{F}(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s)) \right) ds \\
&\quad + \int_{\tau-\delta}^\tau T(\tau-s) \int_0^s \mathbb{M}_g(s, l, z^{\delta,\alpha}(l-r)) dl ds.
\end{aligned}$$

Note that the corresponding solution  $y^{\delta,\alpha}(t) = y(t, \tau - \delta, z(\tau - \delta), u_\alpha)$  of the initial value problem (3.12) at time  $\tau$  is:

$$y^{\delta,\alpha}(\tau) = T(\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau T(\tau-s) \mathbb{B}_\varpi u_\alpha(s) ds.$$

Hence,

$$z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau) = \int_{\tau-\delta}^\tau T(\tau-s) \left( \int_0^s \mathbb{F}(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s)) + \mathbb{M}_g(s, l, z^{\delta,\alpha}(l-r)) dl \right) ds,$$

and from condition (2.7) we obtain that

$$\begin{aligned}
\|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| &\leq \int_{\tau-\delta}^\tau \|T(\tau-s)\| \left( \tilde{a} \|\Phi(s-r)\| + \tilde{b} \right) ds \\
&\quad + \int_{\tau-\delta}^\tau \|T(\tau-s)\| \int_0^s \|\mathbb{M}_g(s, l, z^{\delta,\alpha}(l-r))\| dl ds.
\end{aligned}$$

Observe that  $0 < \delta < r$  and  $\tau - \delta \leq s \leq \tau$ , thus

$$l - r \leq s - r \leq \tau - r < \tau - \delta.$$



Therefore,  $z^{\delta,\alpha}(l-r) = z(l-r)$  and  $z^{\delta,\alpha}(s-r) = z(s-r)$ , implying that for  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\begin{aligned} \|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| &\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left( \tilde{a} \|z(s-r)\| + \tilde{b} \right) ds \\ &\quad + \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \int_0^s \|\mathbb{M}_g(s,l,z(l-r))\| dl ds \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Additionally, for  $0 < \alpha < 1$ , Lemma 3.2 (3.13) yields

$$\|y^{\delta,\alpha}(\tau) - z_1\| < \frac{\epsilon}{2}.$$

Thus,

$$\|z^{\delta,\alpha}(\tau) - z_1\| \leq \|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| + \|y^{\delta,\alpha}(\tau) - z_1\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which completes our proof.

## 5. FINAL REMARKS

We believe that the same technique can be applied for controlling diffusion processes systems involving compact semigroups. In particular, our result can be formulated in a more general setting for the semilinear evolution equation with impulses, delay and memory in a Hilbert space  $\mathcal{Z}$

$$\begin{cases} z' = \mathbb{A}z + \mathbb{B}u + \int_0^t \mathbb{M}_g(t,s,z(s-r))ds + \mathbb{F}(t,z(t-r),u(t)), & z \in \mathcal{Z}, 0 < t \neq t_k, \\ z(s) = \Phi(s), & s \in [-r, 0] \\ z(t_k^+) = z(t_k^-) + \mathbb{I}_k(t_k, z(t_k), u(t_k)), & k = 1, 2, \dots, p, \end{cases}$$

where  $u \in L^2(0, \tau; \mathcal{U})$ ,  $\mathcal{U}$  is another Hilbert space,  $\mathbb{B} : \mathcal{U} \rightarrow \mathcal{Z}$  is a bounded linear operator,  $\mathbb{I}_k, \mathbb{F} : [0, \tau] \times C(-r, 0; \mathcal{Z}) \times \mathcal{U} \rightarrow \mathcal{Z}$ ,  $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{Z} \rightarrow \mathcal{Z}$  is an unbounded linear operator in  $\mathcal{Z}$  that generates a strongly continuous semigroup according to Lemma 2.1 from [13]:

$$(5.16) \quad T(t)z = \sum_{nj=1}^{\infty} e^{A_j t} P_j z, \quad z \in \mathcal{Z}, \quad t \geq 0,$$

where  $\{P_j\}_{j \geq 0}$  is a complete family of orthogonal projections in the Hilbert space  $\mathcal{Z}$  and

$$(5.17) \quad \|F(t, \Phi, u)\|_{\mathcal{Z}} \leq \tilde{a} \|\Phi(-r)\|_{\mathcal{Z}} + \tilde{b}, \quad \forall (t, \Phi, u) \in [0, \tau] \times C(-r, 0; \mathcal{Z}) \times \mathcal{U}.$$

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